V. SUMMARY AND CONCLUSION

The spherical orrery is a useful device for demonstrating and investigating principles of celestial mechanics. The physics of this device is more closely analogous to celestial mechanics than that of an earlier cylindrical orrery in which particles orbit a rod. Phenomena that are easily investigated include Kepler's laws, precession, adiabaticity, molecular drag, and collisions. The use of videotape allows the phenomena to be shown to large audiences. An improved vacuum will allow longer orbital decay times which will facilitate the investigation of resonant perturbations applied for long periods. This type of perturbation is responsible for the Kirkwood gaps in the asteroid belt, much of the fine structure in Saturn's rings, and may also lead to dynamical chaos.

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Thoughts on the magnetic vector potential

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We collect together several ideas that we have found helpful in teaching the magnetic vector potential **A**. We argue that students can be taught to visualize **A** for simple current distributions and to see **A** as something with physical significance beyond its bare definition as the "thing whose curl is **B**."

I. INTRODUCTION

Despite the beautiful symmetry between electric and magnetic fields, the ways in which we teach these two concepts could scarcely be more different. In our introductory physics courses ("freshman physics" in a typical American college), students acquire a reasonable understanding of the electric field **E** and the electrostatic potential ϕ; by contrast, their understanding of the magnetic field **B** is hazy at best, and they probably do not meet the magnetic potential **A** at all. By the end of their next course in electromagnetism ("junior E and M"), students generally have a reasonable understanding of the magnetic field **B** and have met the magnetic potential as a mathematical artifact used to express **B** as $\mathbf{B}=\nabla\times\mathbf{A}$. Nevertheless, they still have almost no idea of what **A** really is, much less any picture of what **A** looks like in even the simplest situations. Even after a graduate course in electrodynamics, many students probably could not say much more about **A** than that it is the vector whose curl is **B**.

In this paper, we focus on the vector potential **A** and argue that there is much that can be said to improve students' understanding of it. Many of the ideas we discuss have appeared before (often in this journal), and some are hinted at in some of the popular textbooks. Nevertheless, it seems clear from the textbooks and from our discussions with numerous colleagues that these ideas are not widely recognized and are certainly not incorporated into most courses in electromagnetism. Given the increasing importance of the vector potential in modern physics (superconductivity, the

Aharonov–Bohm effect, Josephson junctions, SQUIDS, etc.) anything we can say to help our students master the concept seems worth emphasizing.

Perhaps the most obvious difficulty in teaching the vector potential is that it requires a knowledge of vector calculus. One can define the scalar potential ϕ as the potential energy per unit charge and give a remarkably good feeling for ϕ without ever using vector calculus. Typical introductory courses convey a good sense of equipotential surfaces for simple charge distributions and of the rate of change of ϕ as $-\mathbf{E}$ without ever mentioning the equation $\mathbf{E}=-\nabla\phi$. By contrast, it is very hard to teach the vector potential until our students understand the meaning of the curl (as in $\mathbf{B}=\nabla\times\mathbf{A}$). For this reason alone, the vector potential is beyond the reach of almost all freshman physics courses. Obviously, we cannot deny this problem, but we do believe that there is much we can do to improve students' understanding of $\mathbf{A}$ once they do meet it in junior E and M or in a graduate course.

A second obstacle to our students' understanding of the vector potential is the still prevalent view that $\mathbf{A}$ is merely a mathematical fiction whose only role is to express $\mathbf{B}$ as $\nabla\times\mathbf{A}$. Curiously, the founder of our subject, Maxwell himself,[1,2] advocated in 1865 a quite opposite view, which we shall echo, that the vector potential can be seen as a stored momentum per unit charge in much the same way that ϕ is the stored energy per unit charge. Indeed, one of Maxwell's several names for the vector potential was "electromagnetic momentum." An equivalent view, that $\mathbf{A}$ can be seen as the appropriate field momentum per unit charge, was stated by Thomson[3] in 1904 and was forcefully advocated by Konopinski[4] in an article in this journal in 1978. (There is a long history of distinguished articles on related topics in this journal.[5])

The modern view, that $\mathbf{A}$ is an artifact devoid of physical significance seems to originate with Heaviside (in 1886),[6] who described the potentials as "highly artificial quantities," and Hertz (in 1893),[7] who disparaged the components of $\mathbf{A}$ as "magnitudes which serve for calculation only." These views can be found in almost any modern textbook on electromagnetism. Perhaps the strongest statement is due to Rohrlich,[8] who says:

> These functions, known as potentials, have no physical meaning and are introduced solely for the purpose of mathematical simplification of the equations.

Similar statements can be found, for example, in Refs. 9 and 10.

We do not claim that the Maxwell–Thomson view of $\mathbf{A}$ as stored momentum per unit charge is of immense practical value (although we do offer some examples to show the insight that it can contribute). Nonetheless, we do argue that, by giving physical meaning to an otherwise rather abstract notion, this view can help students to feel more at home with and better understand the undeniably important concept of the vector potential.

A third difficulty in teaching the vector potential is that much of its importance appears only later in more advanced subjects which the students of junior E and M have often not studied: In relativity, $\mathbf{A}$ combines with ϕ to form the four-potential $A=(\mathbf{A},\phi/c)$, just as the momentum $\mathbf{p}$ combines with the energy E to form the four-momentum $p=(\mathbf{p},E/c)$. In the Lagrangian mechanics of a charged particle, the generalized, or canonical, momentum turns out to be $\mathbf{p}=m\mathbf{v}+q\mathbf{A}$, and, under the appropriate conditions, it is $\mathbf{p}$ (rather than $m\mathbf{v}$) which is conserved. That is, $q\mathbf{A}$ is the quantity that must be added to $m\mathbf{v}$ to give the "proper" conserved momentum, just as $q\phi$ is the quantity that must be added to $\frac{1}{2}mv^2$ to give the "proper" conserved energy. In quantum theory, $\mathbf{A}$ (as opposed to $\mathbf{B}$) is the fundamental quantity in the Schrödinger equation for a charged particle and in the interactions of quantum electrodynamics.

If your students haven't studied these more advanced subjects, then these arguments for the importance of $\mathbf{A}$ will carry less weight. Nonetheless, some students have studied relativity, Lagrangian mechanics, or even quantum mechanics before meeting the vector potential, and for such students these ideas are well worth exploring. Even if your students have *not* studied these subjects, the arguments can at least be mentioned.

In Sec. II, we describe two ways to help our students visualize, and even calculate, the vector potential for a number of steady current distributions. While most students can readily calculate and visualize the scalar potential ϕ of various charge distributions, very few can do the same for the vector potential of any current distributions. Any tricks to help them do this seem well worth emphasizing. In particular, the formal analogy between $\mu_0\mathbf{J}$ as the source of $\mathbf{B}$ (as in $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$) and $\mathbf{B}$ as the source of $\mathbf{A}$ (as in $\nabla\times\mathbf{A}=\mathbf{B}$) allows one to find the vector potential in several situations by taking advantage of the well-known B fields of certain current distributions. Although this point is mentioned briefly in the fine textbooks[11] of Griffiths and of Barger and Olsson, and is clearly stated in a recent article of Carron,[12] it seems not to be as widely appreciated as it deserves.

In Sec. III, we review the main arguments for the Maxwell–Thomson view that $\mathbf{A}$ is the stored momentum per unit charge, that is, that $\mathbf{A}$ does for momentum what the scalar potential ϕ does for energy. Finally, in Sec. IV, we give some examples of problems that can be solved and perhaps better understood using this way of viewing $\mathbf{A}$.

To conclude this introduction, we need to discuss the consistency of the view of $\mathbf{A}$ as stored momentum with the requirements of gauge invariance. Since $\mathbf{A}$ is not uniquely defined, one is bound to be a bit suspicious of the claim that $\mathbf{A}$ can be interpreted as stored momentum. We shall address this objection at the appropriate points throughout the paper, but it may be helpful to summarize the situation now: First, we note that many familiar physical quantities are not uniquely defined (potential energy, the energy flow vector, the Lagrangian, etc.) but are nevertheless physically significant. In the case of the vector potential, there are many different choices of $\mathbf{A}$, all corresponding to the same electromagnetic fields, and we shall see that each different choice of $\mathbf{A}$ (that is, each different gauge) defines a different generalized momentum $\mathbf{p}=m\mathbf{v}+q\mathbf{A}$. In some gauges the generalized momentum may be conserved and in others it may not. As one might expect, the most convenient choice of gauge is usually one in which $\mathbf{p}$ *is* conserved. Since conservation laws are generally associated with symmetries, this means finding a gauge where $\mathbf{A}$ has the same symmetries as the underlying problem.

II. ON VISUALIZING AND CALCULATING THE VECTOR POTENTIAL

By the time they are in a junior E and M course, most of our students have a reasonable picture of the way the magnetic field $\mathbf{B}$ circles around the current that produces it, and, with the help of Ampere's law, they can calculate the B field

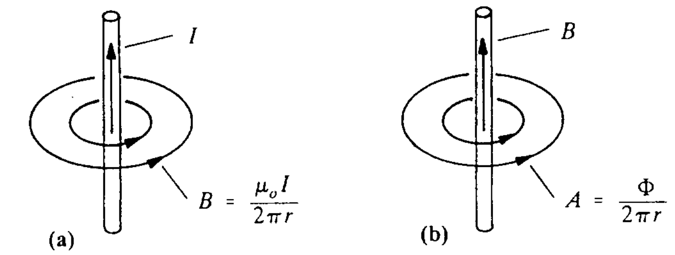

Fig. 1. (a) The B field of a current confined inside a long straight wire circulates around the wire with magnitude given by Ampere's law as $B=\mu_0 I/2\pi r$. (b) The B field of a long solenoid is confined inside the solenoid. Comparing with (a), we can immediately conclude that the vector potential outside the solenoid circulates with magnitude $A=\Phi/2\pi r$.

for several simple current distributions. On the other hand, they usually have very little idea how to visualize or calculate the vector potential for any current distributions.

One approach to finding any magnetostatic **A**, which leans nicely on the students' experience in electrostatics, is to note that, in the gauge where $\nabla\cdot\mathbf{A}=0$, the vector potential **A** satisfies

$$\nabla^2\mathbf{A}=-\mu_0\mathbf{J} \tag{1}$$

(just as the scalar potential satisfies $\nabla^2\phi=-\rho/\epsilon_0$). The solution of this equation is well known to be

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int dV'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{2}$$

(Again, it helps to emphasize the parallel between this and the corresponding result for ϕ.) We mention the well-known result (2) because it makes clear that the contribution of each $\mathbf{J}(\mathbf{r}')$ to $\mathbf{A}(\mathbf{r})$ is in the direction of $\mathbf{J}(\mathbf{r}')$. For example, if **J** has the same direction everywhere (as with the current in a long straight wire), then the same is true of **A**, and **A** has the same direction as **J**. Similarly, Eq. (2) implies that if **J** is axially symmetric and points in circles around its axis of symmetry (as with the current in a circular loop or solenoid), then **A** has these same two properties.

A second way to find **A** is to recognize that (in the gauge with $\nabla\cdot\mathbf{A}=0$) **A** is determined by the two equations

$$\nabla\times\mathbf{A}=\mathbf{B} \quad \text{and} \quad \nabla\cdot\mathbf{A}=0. \tag{3}$$

Comparing these with the two Maxwell equations for **B**,

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \quad \text{and} \quad \nabla\cdot\mathbf{B}=0, \tag{4}$$

we see that **B** can be regarded as the "source" of **A** in just the same way that $\mu_0\mathbf{J}$ is the source of **B**. This analogy means that all the students' hard-won experience using Ampere's law to find **B** for given **J** can be applied immediately to the problem of finding **A** for given **B**. For example, most students in junior E and M are familiar with the way **B** tends to circle around its source current **J**; in just the same way, it follows that the vector potential **A** tends to circle around its corresponding **B**.

A. Example: Vector potential for a solenoid

As a first illustration of this approach to finding **A**, recall that, for a steady current in a long straight cylindrical wire, the B field circulates outside the wire, as shown in Fig. 1(a). Using the analogy between Eqs. (3) and (4), we can immediately find the potential **A** corresponding to the B field of a long cylindrical solenoid, as shown in Fig. 1(b). This field is uniform inside the cylinder and zero outside. Therefore, exactly as **B** circulates around the current **I** in Fig. 1(a), so the vector potential **A** must circulate around **B** in Fig. 1(b).

Quantitatively, the students all know from Ampere's law that the B field outside the current of Fig. 1(a) is

$$B=\frac{\mu_0 I}{2\pi r} \quad \text{[outside wire]}, \tag{5}$$

where I is the total current in the wire. It immediately follows that the vector potential outside the solenoid of Fig. 1(b) must be

$$A=\frac{\Phi}{2\pi r} \quad \text{[outside solenoid]}, \tag{6}$$

where Φ is the total flux of B inside the solenoid. This is surely a most economical and transparent derivation of the vector potential outside a solenoid—a configuration that occurs in the Aharonov–Bohm effect and many other important applications.

We can use the same argument to find the vector potential *inside* the solenoid. If the current in Fig. 1(a) is uniform inside the wire, then we can use Ampere's law, $\oint\mathbf{B}\cdot d\mathbf{r}=\mu_0 I(\text{inside})$, with a circular path of radius r, to show that the B field inside the wire is $B=\mu_0 Ir/2\pi a^2$, where a is the radius of the wire. In exactly the same way, Eq. (3) implies an "Ampere's law" for **A**, namely, $\oint\mathbf{A}\cdot d\mathbf{r}=\Phi(\text{inside})$, and we can see that the vector potential inside the solenoid of Fig. 1(a) circulates around the axis with magnitude

$$A=\frac{\Phi r}{2\pi a^2}=\frac{Br}{2}. \tag{7}$$

If we bear in mind that r in (7) denotes the perpendicular distance out from the axis and we let $a\to\infty$, this result gives the vector potential for a uniform B field:

$$\mathbf{A}=\frac{1}{2}\,\mathbf{B}\times\mathbf{r}, \tag{8}$$

a result of great importance, which appears, for example, in the quantum theory of an atom in a uniform magnetic field (the Zeeman and Paschen–Back effects).

B. Another example: Vector potential for a long straight wire

Figure 2(a) shows a current circulating uniformly around the surface of a long conducting cylinder, that is, a solenoid. The corresponding B field is known (from Ampere's law) to be zero outside the cylinder and uniform, directed along the cylinder, on the inside, as shown in Fig. 2(a). It immediately follows that a B field circulating uniformly around the surface of a cylinder corresponds to a vector potential **A** that is zero outside and uniform inside, as shown in Fig. 2(b). The circulating B field of Fig. 2(b) is produced by a uniform current flowing up a cylinder of radius r and back down a coaxial cylinder of slightly larger radius $r+dr$, as shown in Fig. 2(c). Thus the vector potential of Fig. 2(b) is the potential of a uniform current in the coaxial cable of Fig. 2(c).

It is easy to write down quantitative expressions for the fields involved in this example. The B field inside the sole-

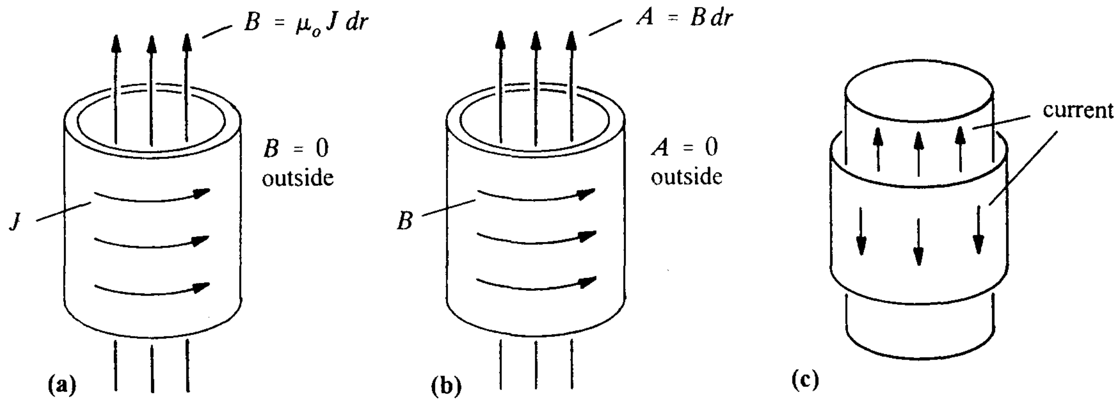

Fig. 2. (a) A current circulating around a long cylinder (or solenoid) produces a B field that is uniform inside the cylinder and zero outside. (b) It immediately follows that a B field circulating around the surface of a cylinder corresponds to a potential **A** that is uniform inside the cylinder and zero outside. (c) The B field of (b) is produced by a current flowing up one cylinder and down a second, coaxial cylinder of slightly larger radius.

noid of Fig. 2(a) is well known to be $\mu_0 nI$, where n is the number of turns per unit length. In terms of the current density J this is

$$B=\mu_0 J\ dr \quad \text{[inside cylinder]}, \tag{9}$$

where dr denotes the thickness of the conducting cylinder. Therefore, the vector potential of Fig. 2(b) is

$$A=B\ dr \tag{10}$$

$$=\frac{\mu_0 I}{2\pi r}\,dr \quad \text{[inside cylinder]}, \tag{11}$$

and $A=0$ outside. Here, dr is the small separation between the two coaxial conductors, and Eq. (11) follows from (10) because the field between the two cylinders is, according to Ampere's law, $B=\mu_0 I/2\pi r$.

From the result (11) we can easily find the vector potential for a single cylindrical wire. Consider, first, two coaxial cylinders at $r=a$ and $r=b$, not necessarily close together. We can regard this arrangement as the superposition of many coaxial pairs, starting at $r=a$ and ending at $r=b$, in which each pair is close together. To find A at any r between a and b, we note that those coaxial pairs inside r contribute nothing to A. Thus the total vector potential at any r between the two cylinders is the integral of (11) from r to b:

$$A(r)=\frac{\mu_0 I}{2\pi}\,[\ln(b)-\ln(r)]. \tag{12}$$

To find the potential for a single wire, we cannot simply let the outer radius b in (12) tend to infinity because the term $\ln(b)$ diverges. However, if we fix b at a value larger than the values of r in which we are interested, then $\ln(b)$ is just a constant, which we can drop, to give

$$A(r)=-\frac{\mu_0 I}{2\pi}\ln(r), \quad [\text{for } a<r<b]. \tag{13}$$

This is the vector potential for two coaxial cylinders of radii a and b. However, we know that the corresponding B field for $a<r<b$ is the same as that outside a single wire of radius a. Since (13) is independent of b, we can now let $b\to\infty$, and we conclude that the vector potential **A** outside a single wire is parallel to the current [the direction predicted in connection with Eq. (2)] and has magnitude given by[13] (13).

One can find other examples of currents for which the known form of **B** for a given $\mu_0\mathbf{J}$ lets one write down **A** for a given **B**. For instance, the vector potential for two antiparallel current sheets and for a toroidal solenoid can both be found easily in this way. (The latter is discussed in detail in Ref. 12.)

III. THE VECTOR POTENTIAL AS "ELECTROMAGNETIC MOMENTUM"

Whenever we define a new concept, we need to say as much as possible—beyond the bare definition—to show our students what the concept really *is*. This is desirable in its own right, but, equally important, it gives the students a context within which to place and understand the new concept. Thus, beyond defining **A** as the "thing whose curl is **B**," we need to say as much as possible about its physical significance. We wish to argue that the Maxwell–Thomson view that **A** is the stored momentum per unit charge supplies this needed physical meaning. Throughout this section we shall consider the motion of a single charge q in an applied electromagnetic field, given by potentials ϕ and **A**.

As a first indication that **A** is at least a candidate for stored momentum per unit charge, we can point out that the units of **A** are precisely those of [momentum/charge]. (Verifying this makes a nice exercise for your students in handling the units of magnetic field.) Another point that is easily made is that in relativity, **A** is related to ϕ exactly as momentum is related to energy. Even students who have not studied relativity formally are almost certainly aware that momentum and energy combine to form the four-momentum $p=(\mathbf{p},E/c)$. Thus we can at least tell them that (as they will learn later) **A** and ϕ combine to form the four-potential $A=(\mathbf{A},\phi/c)$, with **A** in the "momentum slot" and ϕ in the "energy slot."

Perhaps the most compelling argument that **A** is somehow connected with momentum comes from the Lagrangian mechanics of a charged particle in an electromagnetic field. If our students have already learned about Lagrangians, then we can show them that the Lagrangian

$$L = \frac{1}{2} mv^2 - q(\phi - \mathbf{v}\cdot\mathbf{A}) \tag{14}$$

correctly reproduces the equations of motion given by the Lorentz force law. Comparing (14) with the usual form $L=\frac{1}{2}mv^2 - V$, we can identify the quantity $q(\phi - \mathbf{v}\cdot\mathbf{A})$ as a generalized, velocity-dependent potential energy.

From Eq. (14), the canonical momentum is easily seen to have components

$$p_i \equiv \frac{\partial L}{\partial v_i} = mv_i + qA_i. \tag{15}$$

Under the appropriate conditions ($\partial L/\partial x_i = 0$), the component p_i is conserved; thus Eq. (15) shows that $q\mathbf{A}$ is the quantity which, when added to $m\mathbf{v}$, gives the "generalized momentum" that is conserved under the appropriate conditions.

Fortunately, we can derive this result even for those students who have not yet studied Lagrangians. There are in fact two ways to do this, both of which are instructive. Before we describe these two arguments, we remark that the physical predictions based on the Lagrangian (14) are gauge invariant: Different choices for $\mathbf{A}$ lead to different Lagrangians and different canonical momenta; for some choices of $\mathbf{A}$ the canonical momentum (or at least some of its components) may be conserved, while for others it may not. Thus some choices of gauge may be more convenient than others. Nevertheless, the predicted motion of the charged particle and the status of $m\mathbf{v}+q\mathbf{A}$ as the canonical momentum are the same, whatever our choice of gauge.

A. First argument

Our first elementary argument that the quantity $m\mathbf{v}+q\mathbf{A}$ can be interpreted as a generalized momentum starts from Newton's second law, $m\mathbf{a}=\mathbf{F}$, with $\mathbf{F}$ equal to the Lorentz force $q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Let us first recall the argument that the quantity $\frac{1}{2}mv^2 + q\phi$ can be interpreted as a "generalized energy." If we dot Newton's second law with $\mathbf{v}$, we obtain:

$$\frac{d}{dt}\left(\frac{1}{2} mv^2\right) = q\mathbf{v}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B}). \tag{16}$$

If we rewrite $\mathbf{E}$ and $\mathbf{B}$ in terms of the potentials,

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla\times\mathbf{A}, \tag{17}$$

then a moderately challenging exercise in vector calculus shows that Eq. (16) becomes[14]

$$\frac{d}{dt}\left(\frac{1}{2} mv^2 + q\phi\right) = \frac{\partial}{\partial t} q(\phi - \mathbf{v}\cdot\mathbf{A}). \tag{18}$$

Here, the left side is the time rate of change of the "generalized" energy (kinetic+potential). Clearly, this generalized energy is conserved whenever the derivative of the velocity-dependent potential on the right is zero. That is, if both potentials ϕ and $\mathbf{A}$ are independent of time, the generalized energy on the left is conserved.

In an exactly parallel way we can start from Newton's second law, replace the fields by potentials as in Eq. (17), and, after going through some more vector gymnastics, show that[15]

$$\frac{d}{dt}(m\mathbf{v}+q\mathbf{A}) = -\nabla q(\phi - \mathbf{v}\cdot\mathbf{A}). \tag{19}$$

This has the form of Newton's second law ($d\mathbf{p}/dt = -\nabla V$): The left side is the time derivative of the generalized momentum ("kinetic" plus "potential") and it is equal to the gradient of the same velocity-dependent potential as appeared in Eq. (18). Whenever any component of this gradient is zero, the corresponding component of the generalized momentum is conserved.

There are several more points you can make concerning the two equations (18) and (19) if your students are already familiar with Lagrangian mechanics or relativity. First, Eq. (19) is simply Lagrange's equation for the Lagrangian (14). Second, for those reasonably practiced at relativity, it is easy to show that the two equations (18) and (19) combine to form a single covariant equation. Although we chose to write (18) and (19) in nonrelativistic form, both equations are actually correct if we replace the nonrelativistic $\frac{1}{2}mv^2$ and mv with their relativistic counterparts. Next, it is a nice exercise to show that the resulting two equations are equivalent to the single covariant equation[15]

$$\frac{d}{d\tau}(p+qA) = \partial q(U\cdot A), \tag{20}$$

where τ is the proper time, p denotes the relativistic four-momentum $p=\gamma m(\mathbf{v},c)$, A is the four-potential $(\mathbf{A},\phi/c)$, ∂ is the four-dimensional gradient, and U is the four-velocity, $U=\gamma(\mathbf{v},c)$.

B. Second argument

The arguments just given show clearly that $q\mathbf{A}$ is the quantity that must be added to $m\mathbf{v}$ to give the proper, generalized, momentum for a particle in an applied electromagnetic field, but they do not show *why* this is so. An alternative argument, revived by Konopinski[4] but dating back at least to Thomson's book[3] of 1904, fills this gap nicely.

For this second argument, we need to assume that the particle moves nonrelativistically and that the applied fields vary slowly in time. Furthermore, we shall work in the Coulomb gauge, with $\nabla\cdot\mathbf{A}=0$. With these assumptions, we can neglect the magnetic field of the particle compared to its Coulomb field, and the vector potential is given by the familiar Eq. (2).

Recall first that the familiar claim that $q\phi(\mathbf{r})$ is the potential energy of a charge q in an applied field can be justified as follows: We can write down the total field energy (of the particle plus the applied field) as the integral of $\epsilon_0\mathbf{E}^2/2+\mathbf{B}^2/2\mu_0$ over all space. If we write $\mathbf{E}$ as the sum of two terms, one for the particle and one for the applied field, then the contribution to the energy of the cross term $(\mathbf{E}_{\text{part}}\cdot\mathbf{E}_{\text{app}})$ can be identified as the energy of interaction of the particle and field. (Remember that we are neglecting $\mathbf{B}_{\text{part}}$.) The corresponding integral is easily evaluated as

$$\text{(energy of interaction of charge at point } \mathbf{r} \text{ in applied field)} = q\phi(\mathbf{r}), \tag{21}$$

where $\phi(\mathbf{r})$ is the potential of the applied field. This is, of course, just the usual expression for the potential energy of the particle in the field.

In exactly the same way, we can write the total field momentum as the integral of $\epsilon_0\mathbf{E}\times\mathbf{B}$. Again, we can identify the contribution of the cross term $(\mathbf{E}_{\text{part}}\times\mathbf{B}_{\text{app}})$ as the momentum of interaction of the particle and the applied field, and a straightforward manipulation shows it to be

(momentum of interaction of charge at point $\mathbf{r}$ in

$$\text{applied field}) = q\mathbf{A}(\mathbf{r}), \tag{22}$$

This is exactly the result referred to by J. J. Thomson in 1904, when he said[16]

> A simple calculation shows that the whole momentum [of interaction] in the field is equivalent to a momentum $q\mathbf{A}(\mathbf{r})$ at the electrified point, $\mathbf{A}(\mathbf{r})$ being the Vector Potential at $\mathbf{r}$ due to the currents.

This completes the second argument that $q\mathbf{A}$ is the appropriate quantity to add to $m\mathbf{v}$ to give the generalized momentum. It also shows clearly why the generalized momentum $m\mathbf{v}+q\mathbf{A}$ (like the generalized energy $\frac{1}{2}mv^2+q\phi$) is not always conserved. The momentum and energy of the whole system—particle, fields, and sources—are, of course, conserved. However, the quantities $m\mathbf{v}+q\mathbf{A}$ and $\frac{1}{2}mv^2+q\phi$ are just those parts of the momentum and energy directly associated with the particle. Only under certain conditions are these partial quantities conserved. In the case of energy, the condition is, according to Eq. (18), that the potentials be independent of time, a condition that is met in many interesting problems. In the case of the momentum, the condition is, according to Eq. (19), that the gradient of $(\phi-\mathbf{v}\cdot\mathbf{A})$ be zero. As we discuss in Sec. IV, there are situations in which this condition is met (at least for one component of the momentum), and these can be used to give students a sense of the vector potential's tangible physical significance.

Before we go on to the examples of Sec. IV, we should note that the result (22) is *not* gauge invariant. The left side of (22) has the same value for all gauges, whereas the right side does not. This suggests, what is true, that, at least in the quasistatic situations to which Eq. (2) applies, the gauge in which $\mathbf{A}$ is given by (2) is in some ways the most natural gauge to use. Nevertheless, the second example of Sec. IV shows that the generalized momentum $m\mathbf{v}+q\mathbf{A}$ is useful and significant in several different gauges.

IV. EXAMPLES

In this section we describe in some detail three examples of problems where the Maxwell–Thomson view of $\mathbf{A}$ as stored momentum per unit charge gives additional insight into the solutions. Junior-level students are already aware from elementary mechanics that many problems can be solved either by focusing on forces and Newton's second law or by focusing on momentum and energy. In the context of electromagnetism, we can refer to these two approaches as the "force-field approach" and the "momentum-energy approach." The main point of these examples is to show that, in the case of magnetic problems, one can do more than is generally recognized with the momentum-energy approach.

There is some difficulty knowing how to refer to the principal quantities connected with momentum. In the case of energy, $\frac{1}{2}mv^2$ is called the kinetic and $q\phi$ the potential energy, and their sum is often called the total energy (though this term is somewhat elastic and ambiguous). In the case of momentum, $m\mathbf{v}$ is sometimes called the kinetic momentum, and we too shall use this name. The quantity $q\mathbf{A}$ could naturally be called the potential momentum, as was suggested by Konopinski.[4] Although this name is certainly not standard, and one might argue for something more neutral like "stored momentum," we shall follow Konopinski and use "potential momentum." Finally, the sum $m\mathbf{v}+q\mathbf{A}$ can be called the total momentum, but this suffers all the ambiguities of "total energy." In the context of Lagrangian or Hamiltonian mechanics, $m\mathbf{v}+q\mathbf{A}$ is called the canonical momentum,[17] but, since this name seems somewhat specialized, we prefer to use the more neutral "generalized momentum."

A. Example 1: A charge outside a long solenoid

As our first example, we consider the motion of a charge q outside a long solenoid. We imagine q to be stationary and the solenoid to be carrying a steady current, which is switched off at time $t=0$. Variants of this example appear in Feynman's book[18] and Konopinski's article.[4] As was emphasized in the recent article of Johnson, Cragin, and Hodges,[19] and before that by Furry,[20] if the solenoid is an ordinary conducting solenoid, then the charge q will induce charges on the solenoid, and these will produce their own electric field. Although this complication seems not to affect our considerations, it is perhaps worth agreeing to use a nonconducting solenoid to eliminate the difficulty entirely. One way to arrange this is to construct the solenoid from two concentric insulating cylinders, infinitesimally different in radius and coated with uniform and opposite charges. The initial current is created by having either cylinder rotate, and it is switched off by stopping the rotation. To simplify matters, we assume that the current is switched off quickly, so that the charge q does not move appreciably while the current is dropping to zero.

The traditional view of this experiment—in terms of forces and fields—is easily stated: When the current is switched off, the initial magnetic flux Φ_0 in the solenoid drops to zero. This induces an electric field, in the φ direction, which has magnitude (by Faraday's law) $\dot{\Phi}/2\pi r$. (Here, r is the distance of the charge from the axis of the solenoid.) This field delivers a total impulse to the particle, equal to $q\Phi_0/2\pi r$. That is, the charge q moves off in the tangential direction with momentum $q\Phi_0/2\pi r$.

The corresponding account in terms of conservation of momentum is even simpler: Under the conditions assumed, the generalized momentum *is* conserved. While the current is still on, the vector potential outside the solenoid is $A=\Phi_0/2\pi r$ in the tangential direction. Therefore, the charge has potential momentum $qA=q\Phi_0/2\pi r$, but kinetic momentum $mv=0$. When the current is switched off, the potential momentum goes to zero and is converted to kinetic momentum $mv=q\Phi_0/2\pi r$ in the tangential direction.

It is striking to note how precisely Maxwell stated this view of this experiment 120 years ago:[21]

> The vector $\mathbf{A}$ represents in direction and magnitude the time integral [that is, impulse] of the electromagnetic intensity which a particle placed at the point (x,y,z) would experience if the primary current were suddenly stopped.

If the current in the solenoid is switched off slowly, then the problem is more complicated, but the Maxwell–Thomson view is still helpful. We must recognize that the components of the generalized linear momentum are not conserved in this case. [The right side of Eq. (19) is not zero.] However, because of the axial symmetry of the problem, the z component of the generalized *angular* momentum is conserved. (The simplest way to see this is to go over to cylindrical polar coordinates and note that the Lagrangian is independent of φ.) Thus the quantity

$$mr^2\dot{\varphi}+rqA=mr^2\dot{\varphi}+q\Phi/2\pi \tag{23}$$

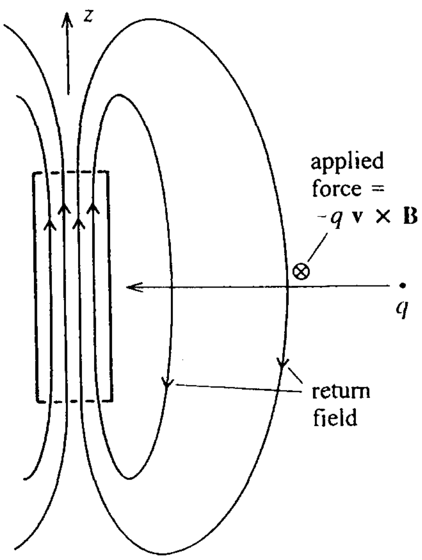

Fig. 3. The field of a long, but finite solenoid is nearly uniform inside, with a much weaker return field outside. As we bring in a charge q from infinity we have to exert a force $-q\mathbf{v}\times\mathbf{B}$. This delivers an angular impulse, as given by Eq. (24).

(kinetic plus potential angular momentum) is conserved.[22] However Φ is switched off, the constancy of (23) lets one write down $\dot{\varphi}$ as a function of r and Φ, and this lets one solve the equation of motion for r once Φ is given. In particular, the final value of the angular momentum $mr^2\dot{\varphi}$ is always just $q\Phi_0/2\pi$.

Another nice feature of this example is that it lets one see that the stored momentum $q\mathbf{A}(\mathbf{r})$ is equal to the impulse we would have to supply to bring the charge q in from infinity to the point $\mathbf{r}$. Our students all know that potential energy is the work we would have to do to bring the charge in from infinity, so it is gratifying to show the corresponding result for the stored momentum (actually *angular* momentum, in this case). An added feature of this particular proof is that it highlights the importance of the return magnetic field for any real solenoid. Figure 3 shows the field of a real (that is, finite length) solenoid. As long as we are concerned with values of r close to the solenoid, we can ignore the return field, but, now that we want to follow the charge in from infinity, the return field is obviously important. As we move the charge in, we must exert a force equal to $-q\mathbf{v}\times\mathbf{B}$, which has magnitude qvB in the φ direction and exerts a torque $\tau = rqvB$ in the z direction. As we move the charge in from infinity to a value of r close to the solenoid, the total angular impulse that we supply is

$$\int \tau\, dt = \int rqvB\, dt = q\int_r^\infty Br'\, dr'. \tag{24}$$

The integral here is $1/2\pi$ times the downward flux of the return field outside the solenoid. Now, the downward flux of the return field equals the upward flux, Φ_0, of the field inside the solenoid. Thus, according to (24), the angular impulse we deliver when bringing up the charge q is $q\Phi_0/2\pi$, which is exactly the stored angular momentum rqA associated with the charge in the field of the solenoid.

B. Example 2: A charge in a uniform magnetic field

As a second example, we consider a charge q in a uniform magnetic field $\mathbf{B}$, which we take to point in the z direction. The motion of such a particle is well known, of course. For example, if its initial position and velocity are in the xy plane, it will remain in the same plane, moving in a circle of radius mv/qB. Nevertheless, one can get additional insight into this motion by considering the conservation of the generalized momentum.

Suppose that the charge is initially at position $\mathbf{r}=\mathbf{R}_0$ and is given a radial velocity v_0. What is the greatest distance $r=R$ to which the charge will move before it starts to move back towards the origin? A slight variant is to suppose the B field is confined to the region $r<R$, and to ask for the minimum speed v_0 for which the charge can escape from the field. Either way, this problem is analogous to throwing a mass up in a gravitational field and asking for its maximum height or minimum escape speed.

In the momentum-energy approach, we would solve this problem as follows: The vector potential for a uniform B field is (Sec. II) $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$, or, in cylindrical polar coordinates, $A=Br/2$ in the φ direction. It is easy to check from Eq. (19) that, with this $\mathbf{A}$, the generalized linear momentum $m\mathbf{v}+q\mathbf{A}$ is not conserved. On the other hand, because of the axial symmetry of the problem, the generalized *angular* momentum about the z axis *is* conserved. That is,

$$r(m\mathbf{v}+q\mathbf{A})_\varphi = mr^2\dot{\varphi}+\tfrac{1}{2}qBr^2 = \text{constant} = \frac{1}{2}\,qBR_0^2, \tag{25}$$

or,

$$mr^2\dot{\varphi} = \frac{1}{2}\,qB(R_0^2-r^2) \quad \text{[cons. of angular momentum]}. \tag{26}$$

Equation (26) shows that the kinetic angular momentum $mr^2\dot{\varphi}$ is initially zero, but as r increases beyond R_0, the kinetic angular momentum must increase in magnitude. (Its sign depends on the sign of the charge q.) To put this more physically, we can say that the equipotential surfaces of the potential angular momentum are cylinders of fixed radius r. Thus, as the charge moves outward, the potential angular momentum changes, so the kinetic angular momentum must also change (in the opposite direction) to conserve the total angular momentum.

The angular momentum equation (26) places no restriction on how large r can grow; it simply relates r and $\dot{\varphi}$. As in the corresponding gravitational problem, the restriction on r comes from energy conservation, which in the present case implies that

$$\frac{1}{2}\,m(\dot{r}^2+r^2\dot{\varphi}^2) = \text{constant} = \frac{1}{2}\,mv_0^2 \quad \text{[cons. of energy]}. \tag{27}$$

This equation shows that, as r and $\dot{\varphi}$ increase, $\dot{r}$ must decrease and eventually vanish. If we denote by R the radius at which $\dot{r}=0$, then it is a straightforward matter to eliminate $\dot{\varphi}$ from Eqs. (26) and (27) to give

$$R = \frac{mv_0}{qB} + \sqrt{\left(\frac{mv_0}{qB}\right)^2 + R_0^2}. \tag{28}$$

This is the maximum radius to which the charge will move. If the field terminates at a given radius R, then it is a simple

exercise to solve Eq. (28) to find the escape speed v_0.

The solution of this problem by the traditional force-field approach is also simple but less insightful. One simply observes that the charge is well known to move in a circle of radius mv_0/qB. (The proof of this result is not entirely trivial, but the result is certainly well known.) This circle is tangent to the initial velocity, and some straightforward geometry shows that the greatest distance of the circle from the origin is given by precisely the result (28).

We have mentioned several times that the concept of the generalized momentum $m\mathbf{v}+q\mathbf{A}$ can be useful in several different gauges, that is, with several different choices of **A**. This is well illustrated by the present example of a charge moving in a uniform magnetic field; specifically, we can take advantage of the freedom to make gauge transformations to give an elegant proof that a charge in a uniform B field does indeed move in a circle of radius mv_0/qB.

The vector potential $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$ that we have used so far for a uniform B field has Cartesian components

$$\mathbf{A}=\frac{B}{2}(-y,x,0) \tag{29}$$

(if we continue to take **B** in the z direction). However, two different choices that give exactly the same value for $\mathbf{B}=\nabla\times\mathbf{A}$ are

$$\mathbf{A}=B(-y,0,0) \tag{30}$$

or

$$\mathbf{A}=B(0,x,0). \tag{31}$$

It is interesting to consider these three different gauges in the light of the symmetries of the original problem: The original uniform B field is symmetric under rotation about the z axis and under arbitrary translations. The choice (29) for **A** exhibits the rotational invariance, but not the translational; the choice (30) is invariant under translations in the x direction (but not the y direction); and the choice (31) is invariant under translations in the y direction (but not the x direction).

Given the symmetries of **A** in the three gauges considered, it is easy to see what will happen to the corresponding generalized momenta. In the gauge (29), it is the φ component of the generalized momentum that is conserved; that is, the z component of the generalized angular momentum is conserved, as we have already noted. In the gauge (30), the x component of the generalized linear momentum is conserved:

$$mv_x+qA_x=mv_x-qBy=\text{constant}. \tag{32}$$

In the same way, if we choose the gauge (31), then it is the y component that is conserved:

$$mv_y+qA_y=mv_y+qBx=\text{constant}. \tag{33}$$

Notice how by making three different choices of **A**, we have learned three different and useful facts.

In particular, we can use the two conditions (32) and (33) to prove that the charged particle moves in a circle. The two constants in Eqs. (32) and (33) can be absorbed into the coordinates x and y (that is, we can shift our origin) to give

$$mv_x=qBy \quad\text{and}\quad mv_y=-qBx. \tag{34}$$

These two equations show that **v** is always perpendicular to **r**. Since $d(r^2)/dt=2\mathbf{v}\cdot\mathbf{r}$, this guarantees that r is constant and that the charge moves in a circle of fixed radius, centered on our new origin. Squaring the two equations (34), we see that the radius of this circle is

$$r=\sqrt{x^2+y^2}=\frac{mv}{qB}. \tag{35}$$

In other words, the momentum-energy approach, with a judicious use of gauge invariance, yields a simple proof that a charge in a uniform magnetic field moves in a circle of radius mv/qB, as well as a more insightful solution of our original problem.

C. Example 3: A uniform B field with a radial E field

Perhaps the most obvious shortcoming of the previous example is that the motion of a charge in a uniform B field is so completely understood that even the most revolutionary approach is unlikely to teach us anything truly new or exciting. Fortunately there are problems where the solution by the traditional force-field approach is hard enough that the point of view advocated here really does help us understand the solution. One such problem is the motion of a charge in a uniform magnetic field *plus* an E field that points radially out (or in) from the axis of the B field.

Consider a uniform B field pointing in the z direction and an infinite charged cylinder (radius R_0) centered on the z axis. This cylinder produces a radial E field of the form $E=\lambda/r$ and potential $\phi=-\lambda\ln r$ (where λ is the linear charge density on the cylinder divided by $2\pi\epsilon_0$). We choose the direction of E such that the electric force on q is radially outward. A relatively straightforward problem is to imagine the charge released from rest at $r=R_0$ and to investigate the nature of the subsequent motion. The energy-momentum solution of this problem is simple and parallels almost exactly that for Example 2 in Sec. IV B. Once again, the axial symmetry implies conservation of the z component of generalized angular momentum. This takes exactly the form of Eq. (26) from Example 2, which shows that, as the electric field accelerates the charge outward, the magnetic field bends it sideways. (The changing potential angular momentum as r increases requires that the kinetic angular momentum changes in the opposite direction.) Conservation of energy takes nearly the same form as before, except that the initial kinetic energy is now zero and there is a potential energy $q\phi(r)$:

$$\begin{aligned}\frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)+q\phi(r)&=\text{constant}\\&=q\phi(R_0),\quad\text{[cons. of energy]}.\end{aligned} \tag{36}$$

Using (26) to eliminate $\dot{\varphi}$ from equation (36), we get an equation for $\dot{r}$, which we can solve in more or less detail depending on what we want to know. One simple application is to suppose there is a second concentric cylinder at radius $R_1>R_0$, in which case our device is approximately the magnetron, of radar fame in World War II.[23] If the E field is too weak, the magnetic field will bend the charge around before it reaches the outer cylinder. Thus, an interesting question to ask is the minimum voltage between the two cylinders that will just accelerate the charge all the way to the outer cylinder. If the charge just reaches the outer cylinder, then $\dot{r}=0$ when $r=R_1$. Making these substitutions into Eq. (36) [with $\dot{\varphi}$ eliminated using (26)] one can easily show that this minimum voltage is

$$V=\phi(R_0)-\phi(R_1)=\frac{qB^2}{8m}\left(R_1-\frac{R_0^2}{R_1}\right)^2. \quad (37)$$

V. CONCLUSION

We have brought together a number of ideas that we have found useful in teaching our students about the magnetic vector potential. In Sec. II, we discussed two ways to visualize and calculate the A field for some common current distributions. In Sec. III, we showed that $q\mathbf{A}$ can be viewed as a potential momentum, very much as $q\phi$ is a potential energy. Finally, in Sec. IV, we illustrated how this interpretation of $q\mathbf{A}$ gives us an alternative, and perhaps more insightful, way to view some problems involving magnetic fields.

[1]J. C. Maxwell, "A dynamical theory of the electromagnetic field," Philos. Trans. **155**, 459–512 (1865). See especially p. 481. The same views are repeated in his book (Ref. 2); see for example Article 590.

[2]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Oxford University, Oxford, 1873), 1st ed. The 3rd edition is more widely available as a Dover reprint (Dover, New York, 1954). Fortunately, the numbering of the articles is the same in all three editions.

[3]J. J. Thomson, *Electricity and Matter* (Charles Scribners, New York, 1904), pp. 32–33.

[4]E. J. Konopinski, "What the electromagnetic vector potential describes," Am. J. Phys. **46**, 499–502 (1978).

[5]See, for example, R. H. Romer, "Angular momentum of static electromagnetic fields," Am. J. Phys. **34**, 772–778 (1966) and *ibid.* **35**, 445–446 (1967); M. G. Calkin, "Linear momentum of quasistatic electromagnetic fields," *ibid.* **34**, 921–925 (1966) and "Liner momentum of the source of a static electromagnetic field," *ibid.* **39**, 513–516 (1971); E. M. Pugh and G. E. Pugh, "Physical significance of the Poynting vector in static fields," *ibid.* **35**, 153–156 (1967). However, the main thrust of these papers was to emphasize the physical reality of the Poynting vector as the field momentum, even when all charges are static; ours is to emphasize the physical significance of the vector potential. There has recently been a revival of interest in the reality (or nonreality) of the Poynting vector; this can be traced from the three "Answers to Question #26 ["Electromagnetic field momentum," R. H. Romer, Am. J. Phys. **63**, 777–779 (1995)]," *ibid.* **64**, 15–17 (1996). Several of the historical facts we mention here came from Y. Gingras, "Comment on 'What the electromagnetic vector potential describes,' " Am. J. Phys. **62**, 84 (1980).

[6]O. Heaviside, "On the self-induction of wires," Philos. Mag. **226**, 118–137 (1886).

[7]H. Hertz, *Electric Waves* (Macmillan, London, 1893, republished by Dover, New York, 1962), p. 196. This book is a collection of articles by Hertz. The article to which we refer was first published in 1890 in the Göttinger Nachrichten.

[8]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965), pp. 65–66.

[9]J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems* (Saunders College, Fort Worth, TX, 1994), 4th ed., p. 202.

[10]D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1989), 2nd ed., p. 228.

[11]Griffiths, Ref. 10, p. 230; V. Barger and M. G. Olsson, *Classical Electricity and Magnetism* (Allyn and Bacon, 1987), 2nd ed., p. 229.

[12]N. J. Carron, "On the fields of a torus and the role of the vector potential," Am. J. Phys. **63**, 717–729 (1995).

[13]It is instructive to derive this same answer directly from Eq. (2). Here, too, one has to be careful to avoid getting an infinite answer. The integral (2) can be evaluated for a wire of length $2L$. If you choose $L \gg r$, you can drop a constant term in $\ln(2L)$ and then let $L\to\infty$ and get the answer (13).

[14]Although much of the vector calculus used in the arguments here is quite tricky, you can make a virtue of this by using it to give your students some much needed exposure to vector calculus (both in homeworks and as worked examples in class). Note also that the partial derivative on the right of (18) acts only on the potential terms ϕ and $\mathbf{A}$.

[15]Again the partial derivative on the right acts only on the potentials.

[16]See Ref. 3, pp. 32–33. We have taken the liberty of translating Thomson's symbols into their modern equivalents. (He uses e for the charge and I for the vector potential.) We have also inserted the words "of interaction" since this is clearly what was being discussed.

[17]In fact, the approach advocated here illustrates that the canonical momentum often *is* the total momentum, just as $\frac{1}{2}mv^2+q\phi$ often is the total energy.

[18]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures in Physics* (Addison-Wesley, Palo Alto, CA, 1965), Vol. II, p. 17–6 and Vol. III, p. 21–5.

[19]F. S. Johnson, B. L. Cragin, and R. R. Hodges, "Electromagnetic momentum density and the Poynting vector in static fields," Am. J. Phys. **62**, 33–41 (1994).

[20]W. H. Furry, "Examples of the momentum distribution in the electromagnetic field and in matter," Am. J. Phys. **37**, 621–636 (1969).

[21]Ref. 2, Article 590.

[22]Just as Eq. (22) shows that $q\mathbf{A}$ is the appropriate part of the field momentum $\int dV\,\epsilon_0\mathbf{E}\times\mathbf{B}$, one can show that $q\mathbf{r}\times\mathbf{A}$ is the corresponding part of the angular momentum $\int dV\,\epsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$.

[23]See, for example, D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics* (Wiley, New York, 1993), 4th ed., pp. 975 and 983.

QUITE A BUG BEAR

... I would not however have you stop in your mathematical course with the works I have mentioned. Every english engineer of much eminence who has entered the profession within the last 20 years has at least a practical acquaintance with the principles of the Differential and Integral calculous. This as you know in college is considered quite a bug bear but I can assure you that you will not find it very difficult provided you have a little patience and commence with the proper book. ...

Joseph Henry, letter to a former student (1845), in *The Papers of Joseph Henry*, edited by Marc Rothenberg (Smithsonian Institution Press, Washington, 1992), Vol. 6, pp. 248–249.